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# GORENSTEIN INJECTIVE AND STRONGLY COTORSION MODULES

BY

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ABSTRACT

By investigating the properties of some special covers and envelopes of modules, we prove that if $R$ is a Gorenstein ring with the injective envelope of ${}_RR$ flat, then a left $R$-module is Gorenstein injective if and only if it is strongly cotorsion, and a right $R$-module is Gorenstein flat if and only if it is strongly torsionfree. As a consequence, we get that for an Auslander–Gorenstein ring $R$, a left $R$-module is Gorenstein injective (resp. flat) if and only if it is strongly cotorsion (resp. torsionfree).

## 1. Introduction

As nice generalizations of injective and flat modules, Enochs et al. introduced in [EJ1] and [EJT] the notions of Gorenstein injective and flat modules. On the other hand, Enochs introduced in [E1] the notion of cotorsion modules. All pure-injective (hence injective) modules are cotorsion. Furthermore, as special cases of cotorsion and torsionfree modules, Xu introduced in [X] the notions of strongly cotorsion and torsionfree modules (over commutative Noetherian rings). Many authors have studied the homological properties of these modules mentioned above, which have turned out to be very useful in characterizing rings; see [E1], [E2], [EH], [EJ1], [EJ2], [EJT], [H], [KS], [MD], [X], and so on. In this paper, we will investigate the relation between Gorenstein injective modules and strongly cotorsion modules. In particular, we will study when

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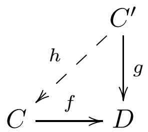
these two kinds of modules coincide as well as when Gorenstein flat modules and strongly torsionfree modules coincide.

In Section 2, we give some terminology and some preliminary results. In Section 3, by investigating the properties of the flat covers, the injective covers and envelopes of modules, we prove that if $R$ is a Gorenstein ring with the injective envelope of ${}_RR$ flat, then a left $R$-module is Gorenstein injective if and only if it is strongly cotorsion, and a right $R$-module is Gorenstein flat if and only if it is strongly torsionfree. As an immediate consequence, we get that a left $R$-module is Gorenstein injective (resp. flat) if and only if it is strongly cotorsion (resp. torsionfree) for an Auslander–Gorenstein ring $R$.

## 2. Preliminaries

Throughout this paper, $R$ is an associative ring with identity and $\mathrm{Mod}\,R$ is the category of left $R$-modules.

*Definition 2.1* ([E1]): Let $\mathscr{C}$ be a full subcategory of $\mathrm{Mod}\,R$. The homomorphism $f : C \to D$ in $\mathrm{Mod}\,R$ with $C \in \mathscr{C}$ is said to be a $\mathscr{C}$**-precover** of $D$ if for any homomorphism $g : C' \to D$ in $\mathrm{Mod}\,R$ with $C' \in \mathscr{C}$, there exists a homomorphism $h : C' \to C$ such that the following diagram commutes:




The homomorphism $f : C \to D$ is said to be **right minimal** if an endomorphism $h : C \to C$ is an automorphism whenever $f = fh$. A $\mathscr{C}$-precover $f : C \to D$ is called a $\mathscr{C}$**-cover** if $f$ is right minimal; $\mathscr{C}$ is called **covering** if every module in $\mathrm{Mod}\,R$ has a $\mathscr{C}$-cover. Dually, the notions of a $\mathscr{C}$**-preenvelope**, a **left minimal homomorphism** and **a** $\mathscr{C}$**-envelope** are defined.

For a full subcategory $\mathscr{C}$ of $\mathrm{Mod}\,R$, we denote by

$$\mathscr{C}^{\perp_1} = \{X \in \mathrm{Mod}\,R \mid \mathrm{Ext}^1_R(C, X) = 0 \text{ for any } C \in \mathscr{C}\}.$$

The following useful result is usually called the Wakamatsu's lemma.

Lemma 2.2 ([EJ2, Corollary 7.2.3]): *Let $\mathscr{C}$ be a full subcategory of* $\operatorname{Mod} R^{op}$ *closed under extensions. If $f : C \to D$ is a $\mathscr{C}$-cover of a module $D$ in* $\operatorname{Mod} R^{op}$, *then* $\operatorname{Ker} f \in \mathscr{C}^{\perp_1}$.

For a module $M \in \operatorname{Mod} R$, we denote the flat, injective and projective dimensions of $M$ by $\operatorname{fd}_R M$, $\operatorname{id}_R M$ and $\operatorname{pd}_R M$, respectively. The following lemma will be used frequently in next section.

Lemma 2.3:

(1) ([I, Proposition 1]) *For a right Noetherian ring $R$,*

$$\operatorname{id}_{R^{op}} R = \sup\{\operatorname{fd}_R I \mid I \in \operatorname{Mod} R \textit{ is injective }\}.$$

(2) ([DC, Theorem 3.8]) *For a left Noetherian ring $R$,*

$$\operatorname{id}_R R = \sup\{\operatorname{id}_R M \mid M \in \operatorname{Mod} R \textit{ with } \operatorname{fd}_R M < \infty\}.$$

*Definition 2.4* ([X, EH]): A module $M \in \operatorname{Mod} R$ is called **cotorsion** if $\operatorname{Ext}^1_R(F, M) = 0$ for any flat left $R$-module $F$; and $M$ is called **strongly cotorsion** if $\operatorname{Ext}^1_R(X, M) = 0$ for any $X \in \operatorname{Mod} R$ with finite flat dimension. A module $N \in \operatorname{Mod} R^{op}$ is called **strongly torsionfree** if $\operatorname{Tor}^R_1(N, X) = 0$ for any $X \in \operatorname{Mod} R$ with finite flat dimension.

Enochs and Huang introduced in [EH] the notion of $n$-cotorsion modules as follows.

*Definition 2.5:* For a non-negative integer $n$, a module $M \in \operatorname{Mod} R$ is called **$n$-cotorsion** if $\operatorname{Ext}^1_R(X, M) = 0$ for any $X \in \operatorname{Mod} R$ with flat dimension at most $n$.

It is trivial that a module $M$ is cotorsion if and only if it is 0-cotorsion, and $M$ is strongly cotorsion if and only if it is $n$-cotorsion for all $n$.

As a generalization of injective modules, the notion of Gorenstein injective modules was introduced by Enochs and Jenda in [EJ1].

*Definition 2.6* ([EJ1]): A module $M \in \operatorname{Mod} R$ is called **Gorenstein injective** if there exists an exact sequence:

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in $\operatorname{Mod} R$ with all terms injective, such that $M = \operatorname{Im}(I_0 \to I^0)$ and the sequence is still exact after applying the functor $\operatorname{Hom}_R(I, -)$ for any injective left $R$-module $I$.

Recall that a left and right Noetherian ring $R$ is called **Gorenstein** if $\operatorname{id}_R R = \operatorname{id}_{R^{op}} R < \infty$.

LEMMA 2.7: *Let $R$ be a Gorenstein ring with $\operatorname{id}_R R = \operatorname{id}_{R^{op}} R = n$. If $n \geq 1$, a module $G \in \operatorname{Mod} R$ is Gorenstein injective if and only if there exists an exact sequence*

$$I_{n-1} \to \cdots \to I_1 \to I_0 \to G \to 0$$

*in $\operatorname{Mod} R$ with all $I_i$ injective. If $n = 0$, then every module in $\operatorname{Mod} R$ is Gorenstein injective.*

*Proof.* We first prove the former assertion. The necessity is trivial. We only need to prove the sufficiency. Let $K_n = \operatorname{Ker}(I_{n-1} \to I_{n-2})$ (note: $I_{-1} = G$) and $I$ be an injective module in $\operatorname{Mod} R$. Because $R$ is a Gorenstein ring with $\operatorname{id}_R R = \operatorname{id}_{R^{op}} R = n$, $\operatorname{pd}_R I \leq n$ by [I, Theorem 2]. So $\operatorname{Ext}_R^i(I, G) \cong \operatorname{Ext}_R^{n+i}(I, K_n) = 0$ for any $i \geq 1$. Thus $G$ is Gorenstein injective by [EJ2, Corollary 11.2.2]. The latter assertion also follows from [EJ2, Corollary 11.2.2]. ∎

As a generalization of flat modules, the notion of Gorenstein flat modules was introduced by Enochs, Jenda and Torrecillas in [EJT].

*Definition 2.8* ([EJT]): A module $N \in \operatorname{Mod} R^{op}$ is called **Gorenstein flat** if there exists an exact sequence:

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

in $\operatorname{Mod} R^{op}$ with all terms flat, such that $N = \operatorname{Im}(F_0 \to F^0)$ and the sequence is still exact after applying the functor $- \bigotimes_R I$ for any injective left $R$-module $I$.

We denote by $(-)^+ = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

LEMMA 2.9: *For a module $N \in \operatorname{Mod} R^{op}$, we have:*

1. *$N$ is strongly torsionfree if and only if $N^+$ is strongly cotorsion.*
2. *If $R$ is a Gorenstein ring, then $N$ is Gorenstein flat if and only if $N^+$ is Gorenstein injective.*

*Proof.* (1) By [CE, Chapter VI, Proposition 5.1], we have

$$[\mathrm{Tor}_1^R(N,A)]^+ \cong \mathrm{Ext}_R^1(A,N^+)$$

for any $A \in \mathrm{Mod}\, R$. Then the assertion follows easily.

(2) See [EJ2, Theorem 10.3.8]. ∎

## 3. Main results

For a full subcategory $\mathscr{C}$ of $\mathrm{Mod}\, R$, we denote the $\mathscr{C}$-cover of a module $M \in \mathrm{Mod}\, R$ by $\mathscr{C}_0(M)$ if it exists. The following result gives a criterion for judging when a $\mathscr{C}$-cover of a module is in another subcategory $\mathscr{D}$ of $\mathrm{Mod}\, R$, which is indebted to [KS, Theorem 3.1].

Theorem 3.1: *Let $M \in \mathrm{Mod}\, R$ and $\mathscr{C}$ be a covering subcategory of $\mathrm{Mod}\, R$ such that either $\mathscr{C}$ is closed under extensions or $\mathscr{C} \subseteq \mathscr{C}^{\perp_1}$, and such that there exists an epic $\mathscr{C}$-cover of $M$, and let $\mathscr{D}$ be a subcategory of $\mathrm{Mod}\, R$ closed under direct summands such that there exists a $\mathscr{D}$-precover of $M$ and $\mathscr{D} \subseteq \mathscr{D}^{\perp_1} \bigcap \mathscr{C}^{\perp_1}$. If the $\mathscr{C}$-cover of any $\mathscr{D}$-precover of $M$ is in $\mathscr{D}$, then the following statements are equivalent.*

1. *$\mathscr{C}_0(M) \in \mathscr{D}$.*
2. *Every $\mathscr{D}$-precover $D \xrightarrow{f} M$ of $M$ is epic and $\mathrm{Ker}\, f \in \mathscr{C}^{\perp_1}$.*
3. *There exists a $\mathscr{D}$-precover $D \xrightarrow{f} M$ of $M$ such that $f$ is epic and $\mathrm{Ker}\, f \in \mathscr{C}^{\perp_1}$.*

*Proof.* (1) $\Rightarrow$ (2) Because there exists an epic $\mathscr{C}$-cover of $M$ and $\mathscr{C}_0(M) \in \mathscr{D}$, every $\mathscr{D}$-precover $D \xrightarrow{f} M$ of $M$ is epic. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\, f & \longrightarrow & A & \longrightarrow & \mathscr{C}_0(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\, f & \longrightarrow & D & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

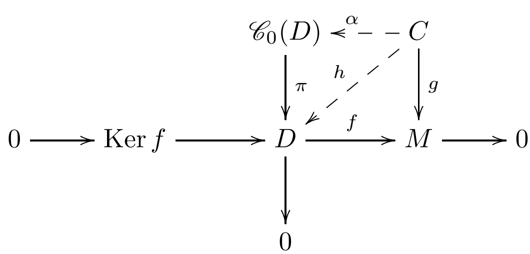
where $K = \mathrm{Ker}(\mathscr{C}_0(M) \to M)$. Because $D \xrightarrow{f} M$ is a $\mathscr{D}$-precover of $M$ and $\mathscr{D} \subseteq \mathscr{D}^{\perp_1}$, $\mathrm{Ker}\, f \in \mathscr{D}^{\perp_1}$. In particular, $\mathrm{Ext}^1_R(\mathscr{C}_0(M), \mathrm{Ker}\, f) = 0$. So the middle row in the above diagram splits, which implies that $\mathrm{Ker}\, f$ is isomorphic to a direct summand of $A$. On the other hand, note that $K \in \mathscr{C}^{\perp_1}$ by Lemma 2.2, and $D \in \mathscr{C}^{\perp_1}$ by assumption. Thus by the exactness of the middle column in the above diagram, $A \in \mathscr{C}^{\perp_1}$ and $\mathrm{Ker}\, f \in \mathscr{C}^{\perp_1}$.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Assume that there exists an exact sequence

$$0 \to \mathrm{Ker}\, f \to D \xrightarrow{f} M \to 0$$

in $\mathrm{Mod}\, R$ such that $D \xrightarrow{f} M$ is a $\mathscr{D}$-precover of $M$ and $\mathrm{Ker}\, f \in \mathscr{C}^{\perp_1}$. Let $g : C \to M$ be a homomorphism in $\mathrm{Mod}\, R$ with $C \in \mathscr{C}$. Consider the following diagram:




where $\pi : \mathscr{C}_0(D) \to D$ is the $\mathscr{C}$-cover of $D$. Because $\mathrm{Ker}\, f \subseteq \mathscr{C}^{\perp_1}$, there exists a homomorphism $h : C \to D$ such that $g = fh$. Then there exists a homomorphism $\alpha : C \to \mathscr{C}_0(D)$ such that $h = \pi\alpha$. Thus we have $g = fh = (f\pi)\alpha$, which implies that $f\pi : \mathscr{C}_0(D) \to M$ is a $\mathscr{C}$-precover of $M$. It follows that $\mathscr{C}_0(M)$ is isomorphic to a direct summand of $\mathscr{C}_0(D)$. On the other hand, $\mathscr{C}_0(D) \in \mathscr{D}$ and $\mathscr{D}$ is closed under direct summands by assumption. Thus we have that $\mathscr{C}_0(M) \in \mathscr{D}$. ∎

For a non-negative integer $n$, we use $\mathscr{F}_n(R)$ to denote the subcategory of $\mathrm{Mod}\, R$ consisting of modules with flat dimension at most $n$. Bican, El Bashir and Enochs proved in [BEE] that every module has a flat cover for any ring. Mao and Ding generalized this result and proved in [MD, Theorem 3.4] that every module in $\mathrm{Mod}\, R$ has an $\mathscr{F}_n(R)$-cover. In particular, an $\mathscr{F}_0(R)$-cover is just a flat cover. We denote the $\mathscr{F}_n(R)$-cover of a module $M \in \mathrm{Mod}\, R$ by $F_n(M)$. On the other hand, by [E1, Theorem 2.1], we have that every module in $\operatorname{Mod} R$ has an injective cover if and only if $R$ is a left Noetherian ring. The following result is an immediate consequence of Theorem 3.1.

Corollary 3.2: *Let $R$ be a left Noetherian ring and $M \in \operatorname{Mod} R$. If the $\mathscr{F}_n(R)$-cover of any injective precover of $M$ is injective, where $n$ is a non-negative integer, then the following statements are equivalent.*

(1) *$F_n(M)$ is injective.*
(2) *Every injective precover $E \xrightarrow{f} M$ of $M$ is epic and $\operatorname{Ker} f$ is $n$-cotorsion.*
(3) *There exists an injective precover $E \xrightarrow{f} M$ of $M$ such that $f$ is epic and $\operatorname{Ker} f$ is $n$-cotorsion.*

For a module $M \in \operatorname{Mod} R$, we denote the injective envelope and the injective cover (if it exists) of $M$ by $E^0(M)$ and $E_0(M)$, respectively. Putting $n = 0$ in Corollary 3.2, we get the following result, which is a non-commutative analog of [KS, Theorem 3.1].

Corollary 3.3: *Let $R$ be a left and right Noetherian ring with $E^0({}_RR)$ flat. Then the following statements are equivalent for a module $M \in \operatorname{Mod} R$.*

(1) *$F_0(M)$ is injective.*
(2) *Every injective precover $E \xrightarrow{f} M$ of $M$ is epic and $\operatorname{Ker} f$ is cotorsion.*
(3) *There exists an injective precover $E \xrightarrow{f} M$ of $M$ such that $f$ is epic and $\operatorname{Ker} f$ is cotorsion.*

*Proof.* By [EH, Theorem 4.5], $E^0({}_RR)$ is flat if and only if $F_0(E)$ is injective for any injective left $R$-module $E$. So the assertion follows from Corollary 3.2. ∎

The following result is a non-commutative analog of [X, Lemma 4.1].

Proposition 3.4: *Let $R$ be a right Noetherian ring with $\operatorname{id}_{R^{op}} R < \infty$ and let $C \in \operatorname{Mod} R$ be strongly cotorsion. Then there exists an exact sequence*

$$0 \to K \to E_0(C) \xrightarrow{f} C \to 0$$

*in $\operatorname{Mod} R$ such that $K$ is strongly cotorsion and $\operatorname{Ext}^1_R(I, K) = 0$ for any injective left $R$-module $I$.*

*Proof.* Let $C \in \operatorname{Mod} R$ be strongly cotorsion. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & F_0(C) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & E^0(F_0(C)) & \longrightarrow & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $H = \operatorname{Ker}(F_0(C) \to C)$ and $X = \operatorname{Coker}(F_0(C) \to E^0(F_0(C)))$. Because $\operatorname{id}_{R^{op}} R < \infty$, $\operatorname{fd}_R E^0(F_0(C)) < \infty$ by Lemma 2.3(1). So $\operatorname{fd}_R X < \infty$ by the exactness of the middle column in the above diagram, and hence $\operatorname{Ext}_R^1(X, C) = 0$. It yields that the rightmost column $0 \to C \to D \to X \to 0$ in the above diagram splits. Then we get an epimorphism $E^0(F_0(C)) \to C$. So the injective cover $f : E_0(C) \to C$ of $C$ is epic. By Lemma 2.2, $\operatorname{Ext}_R^1(I, K) = 0$ for any injective left $R$-module $I$, where $K = \operatorname{Ker} f$.

Let $A \in \operatorname{Mod} R$ with $\operatorname{fd}_R A < \infty$. Then from the exact sequence

$$0 = \operatorname{Ext}_R^1(A, C) \to \operatorname{Ext}_R^2(A, K) \to \operatorname{Ext}_R^2(A, E_0(C)) = 0$$

we get that $\operatorname{Ext}_R^2(A, K) = 0$. Consider the following exact sequence:

$$0 \to A \to E^0(A) \to B \to 0,$$

where $B = \operatorname{Coker}(A \to E^0(A))$. By Lemma 2.3(1), $\operatorname{fd}_R E^0(A) < \infty$. So $\operatorname{fd}_R B < \infty$ and hence $\operatorname{Ext}_R^2(B, K) = 0$ by the above argument. Then from the exact sequence

$$0 = \operatorname{Ext}_R^1(E^0(A), K) \to \operatorname{Ext}_R^1(A, K) \to \operatorname{Ext}_R^2(B, K) = 0$$

we get that $\operatorname{Ext}_R^1(A, K) = 0$ and $K$ is strongly cotorsion. ∎

For a left Noetheran ring $R$ and a module $M \in \operatorname{Mod} R$, by [E1, Theorem 2.1] we have the following complex:

$$\cdots \to E_i(M) \to \cdots \to E_1(M) \to E_0(M) \to M$$

in $\operatorname{Mod} R$ such that $E_0(M) \to M$ is the injective cover of $M$ and $E_{i+1}(M) \to K_i$ is the injective cover of $K_i$ for any $i \geq 0$, where $K_i = \operatorname{Ker}(E_i(M) \to E_{i-1}(M))$ (note: $E_{-1}(M) = M$).

The next corollary finishes the proof of the main result in one direction.

Corollary 3.5: *For a Gorenstein ring $R$, a strongly cotorsion left $R$-module is Gorenstein injective.*

*Proof.* Let $C \in \operatorname{Mod} R$ be strongly cotorsion. By Proposition 3.4, there exists an exact sequence

$$0 \to K_0 \to E_0(C) \to C \to 0$$

in $\operatorname{Mod} R$ such that $K_0$ is strongly cotorsion and $\operatorname{Ext}^1_R(I, K_0) = 0$ for any injective left $R$-module $I$. Then it is easy to see that we can get an exact sequence

$$\cdots \to E_i(C) \to \cdots \to E_1(C) \to E_0(C) \to C \to 0$$

in $\operatorname{Mod} R$ such that $\operatorname{Ext}^1_R(I, K_i) = 0$ for any injective left $R$-module $I$ and $i \geq 0$, where $K_i = \operatorname{Im}(E_{i+1}(C) \to E_i(C))$. It follows from Lemma 2.7 that $C$ is Gorenstein injective. ∎

By Corollary 3.5, we have the following

Corollary 3.6:

(1) *If $R$ is a Gorenstein ring, then a left $R$-module is injective if and only if it is strongly cotorsion with finite injective dimension.*
(2) *If $R$ is a right Noetherian ring with $\operatorname{id}_{R^{op}} R < \infty$, then a left $R$-module is injective if it is strongly cotorsion with finite flat dimension.*

*Proof.* (1) The necessity is trivial. We only need to prove the sufficiency. Let $C \in \operatorname{Mod} R$ be strongly cotorsion. Then $C$ is Gorenstein injective by Corollary 3.5. So if $\operatorname{id}_R C < \infty$, then $C$ is injective.

(2) Let $C \in \operatorname{Mod} R$ be strongly cotorsion with $\operatorname{fd}_R C < \infty$. Then $\operatorname{fd}_R E^0(C) < \infty$ by Lemma 2.3(1). Consider the following exact sequence:

$$0 \to C \to E^0(C) \to X \to 0.$$

Then $\operatorname{fd}_R X < \infty$. So $\operatorname{Ext}_R^1(X, C) = 0$ and this exact sequence splits. Thus $C$ is injective. ∎

The following result is a dual version of Corollary 3.6.

Corollary 3.7: (1) *If $R$ is a Gorenstein ring, then a right $R$-module is flat if and only if it is strongly torsionfree with finite flat dimension.*

(2) *If $R$ is a right Noetherian ring with $\operatorname{id}_{R^{op}} R < \infty$, then a right $R$-module is flat if it is strongly torsionfree with finite injective dimension.*

*Proof.* (1) The necessity is trivial. We only need to prove the sufficiency. Let $T \in \operatorname{Mod} R^{op}$ be strongly torsionfree with $\operatorname{fd}_{R^{op}} T < \infty$. Then $T^+$ is strongly cotorsion with $\operatorname{id}_R T^+ < \infty$ by Lemma 2.9(1) and [F, Theorem 2.1]. So $T^+$ is injective by Corollary 3.6(1), and hence $T$ is flat by [F, Theorem 2.1].

(2) Let $T \in \operatorname{Mod} R^{op}$ be strongly torsionfree with $\operatorname{id}_{R^{op}} T < \infty$. Then $T^+$ is strongly cotorsion with $\operatorname{fd}_R T^+ < \infty$ by Lemma 2.9(1) and [F, Theorem 2.2]. So $T^+$ is injective by Corollary 3.6(2), and hence $T$ is flat by [F, Theorem 2.1]. ∎

We continue with some propositions leading to the proof of the main result in the other direction. We denote by $\mathscr{I}(R)$ the full subcategory of $\operatorname{Mod} R$ consisting of modules with finite injective dimension, and by

$$\mathscr{I}(R)^{\perp_n} = \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^i(K, M) = 0 \text{ for any } K \in \mathscr{I}(R) \text{ and } 1 \leq i \leq n\}$$

for a positive integer $n$.

Proposition 3.8: *Let $R$ be a left and right Noetherian ring with $\operatorname{id}_R R < \infty$ and let $M \in \mathscr{I}(R)^{\perp_1}$. If $E^0({}_R R)$ is flat, then $F_0(M)$ is injective.*

*Proof.* Let $M \in \mathscr{I}(R)^{\perp_1}$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & F_0(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & E^0(F_0(M)) & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $A = \mathrm{Ker}(F_0(M) \to M)$ and $K = \mathrm{Coker}(F_0(M) \to E^0(F_0(M)))$. Because $E^0({}_RR)$ is flat by assumption, $E^0(F_0(M))$ is flat by [EH, Theorem 4.5]. On the other hand, by Lemma 2.2, we have that $\mathrm{Ext}^1_R(F, A) = 0$ for any flat left $R$-module $F$. So by the exactness of the middle row in the above diagram, $E^0(F_0(M))$ is a flat precover of $X$.

Because $\mathrm{id}_R R < \infty$, $\mathrm{id}_R F_0(M) < \infty$ by Lemma 2.3(2). So $\mathrm{id}_R K < \infty$ and $\mathrm{Ext}^1_R(K, M) = 0$, which implies that the rightmost column

$$0 \to M \to X \to K \to 0$$

in the above diagram splits. It follows that $M$ is isomorphic to a direct summand of $X$; then $E^0(F_0(M))$ is a flat precover also of $M$ and hence $F_0(M)$ is isomorphic to a direct summand of $E^0(F_0(M))$. Thus $F_0(M)$ is injective. ∎

We denote by $\mathscr{I}(R)^{\perp} = \bigcap_{n\geq 1} \mathscr{I}(R)^{\perp_n}$. By Proposition 3.8, we get the following

Proposition 3.9: *Let $R$ be a left and right Noetherian ring with $\mathrm{id}_R R < \infty$ and let $M \in \mathscr{I}(R)^{\perp}$. If $E^0({}_RR)$ is flat, then $M$ is strongly cotorsion.*

*Proof.* Let $M \in \mathscr{I}(R)^{\perp}$. By Proposition 3.8, $F_0(M)$ is injective. Then by Corollary 3.3, there exists an exact sequence:

$$0 \to M_1 \to E_0(M) \to M \to 0 \tag{1}$$

in $\mathrm{Mod}\, R$ with $M_1$ cotorsion. Because $E_0(M) \to M$ is the injective cover of $M$, $\mathrm{Ext}_R^1(I, M_1) = 0$ for any injective left $R$-module $I$ by Lemma 2.2. Then $\mathrm{Ext}_R^i(I, M_1) = 0$ for any injective left $R$-module $I$ and $i \geq 1$ by the exact sequence (1). It follows easily that $M_1 \in \mathscr{I}(R)^{\perp}$.

Let $X \in \mathrm{Mod}\, R$ with $\mathrm{fd}_R\, X < \infty$. Because $\mathrm{id}_R\, R < \infty$ by assumption, $\mathrm{id}_R\, X < \infty$ by Lemma 2.3(2). So $\mathrm{Ext}_R^i(X, M_1) = 0$ for any $i \geq 1$. Then from the exact sequence (1) we get that $\mathrm{Ext}_R^1(X, M) \cong \mathrm{Ext}_R^2(X, M_1) = 0$ and $M$ is strongly cotorsion. ∎

We are now in a position to state the main result.

Theorem 3.10: *Let $R$ be a Gorenstein ring with $E^0({}_RR)$ flat. Then we have:*

1. *A left $R$-module is Gorenstein injective if and only if it is strongly cotorsion.*
2. *A right $R$-module is Gorenstein flat if and only if it is strongly torsionfree.*

*Proof.* (1) By [H, Theorem 2.22], we have that $M \in \mathscr{I}(R)^{\perp}$ if $M \in \mathrm{Mod}\, R$ is Gorenstein injective. So the assertion follows from Corollary 3.5 and Proposition 3.9.

(2) By Lemma 2.9 and (1), we have that a module $N \in \mathrm{Mod}\, R^{op}$ is Gorenstein flat if and only if $N^+$ is Gorenstein injective, if and only if $N^+$ is strongly cotorsion, and if and only if $N$ is strongly torsionfree. ∎

Let

$$0 \to {}_RR \to E^0({}_RR) \to E^1({}_RR) \to \cdots \to E^i({}_RR) \to \cdots$$

be a minimal injective coresolution of ${}_RR$. Recall from [Bj] that a left and right Noetherian ring $R$ is said to satisfy the **Auslander condition** if $\mathrm{fd}_R\, E^i({}_RR) \leq i$ for any $i \geq 0$, and $R$ is called **Auslander–Gorenstein** if it is Gorenstein and satisfies the Auslander condition. It is well known that the notion of the Auslander condition is left-right symmetric ([FGR, Theorem 3.7]). By [B, Fundamental Theorem], we have that a commutative Gorenstein ring is Auslander–Gorenstein. By Theorem 3.10, we immediately have the following

Corollary 3.11: *For an Auslander–Gorenstein ring $R$, a left $R$-module is Gorenstein injective (resp. flat) if and only if it is strongly cotorsion (resp. torsionfree). In particular, for a commutative Gorenstein ring $R$, an $R$-module is*

*Gorenstein injective (resp. flat) if and only if it is strongly cotorsion (resp. torsionfree).*

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